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Existence of a positive solution to systems of differential equations of fractional order

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ABSTRACT

In this paper, we consider a system of (continuous) fractional boundary value problems given by

$$\begin{cases} -D_{0+}^{\nu_1} y_1(t) = \lambda_1 a_1(t) f\left(y_1(t), y_2(t)\right), \\ -D_{0+}^{\nu_2} y_2(t) = \lambda_2 a_2(t) g\left(y_1(t), y_2(t)\right), \end{cases}$$

where $\nu_1, \nu_2 \in (n-1, n]$ for $n > 3$ and $n \in \mathbb{N}$, subject either to the boundary conditions $y_1^{(i)}(0) = 0 = y_2^{(i)}(0)$, for $0 \le i \le n-2$, and $\left[D_{0+}^{\alpha} y_1(t)\right]_{t=1} = 0 = \left[D_{0+}^{\alpha} y_2(t)\right]_{t=1}$, for $1 \le \alpha \le n-2$, or $y_1^{(i)}(0) = 0 = y_2^{(i)}(0)$, for $0 \le i \le n-2$, and $\left[D_{0+}^{\alpha} y_1(t)\right]_{t=1} = \phi_1(y)$, for $1 \le \alpha \le n-2$, and $\left[D_{0+}^{\alpha} y_2(t)\right]_{t=1} = \phi_2(y)$, for $1 \le \alpha \le n-2$. In the latter case, the continuous functionals $\phi_1, \phi_2 : \mathcal{C}([0,1]) \to \mathbb{R}$ represent nonlocal boundary conditions. We provide conditions on the nonlinearities f and g, the nonlocal functionals ϕ_1 and ϕ_2, and the eigenvalues λ_1 and λ_2 such that the system exhibits at least one positive solution. Our results here generalize some recent results on both scalar fractional boundary value problems and systems of fractional boundary value problems, and we provide two explicit numerical examples to illustrate the generalizations that our results afford.

1. Introduction

In this paper we consider a system of nonlinear differential equations of fractional order having the form

$$\begin{cases} -D_{0+}^{\nu_1} y_1(t) = \lambda_1 a_1(t) f\left(y_1(t), y_2(t)\right), \\ -D_{0+}^{\nu_2} y_2(t) = \lambda_2 a_2(t) g\left(y_1(t), y_2(t)\right), \end{cases} \tag{1.1}$$

where $t \in (0,1)$, $\nu_1, \nu_2 \in (n-1, n]$, and $\lambda_1, \lambda_2 > 0$, subject to a couple of different sets of boundary conditions. In particular, we first consider problem (1.1) subject to

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0), \quad 0 \le i \le n-2, \tag{1.2}$$

$$\left[D_{0+}^{\alpha} y_1(t)\right]_{t=1} = 0 = \left[D_{0+}^{\alpha} y_2(t)\right]_{t=1}, \quad 1 \le \alpha \le n-2, \tag{1.3}$$

where $y^{(i)}$ in boundary condition (1.2) represents the i-th (ordinary) derivative of y. We then consider the case in which the boundary conditions are changed to

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0), \quad 0 \le i \le n-2, \tag{1.4}$$

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$$\left[D_{0+}^{\alpha}y_1(t)\right]_{t=1} = \phi_1(y), \quad 1 \le \alpha \le n-2, \tag{1.5}$$

$$\left[D_{0+}^{\alpha}y_2(t)\right]_{t=1} = \phi_2(y), \quad 1 \le \alpha \le n-2, \tag{1.6}$$

where $\phi_1, \phi_2 : \mathcal{C}([0, 1]) \to \mathbb{R}$ are continuous functionals, where the notation $\mathcal{C}([0, 1])$ means the set of continuous, real-valued functions on the unit interval $[0, 1]$; even though (1.3) and (1.5)–(1.6) lose physical meaning when $\alpha \notin \mathbb{N}$, they are still mathematically meaningful. We also consider these boundary conditions in the special case in which $\lambda_1 = \lambda_2 = 1$. Note that in (1.1), (1.3), (1.5), (1.6), and, in fact throughout this work, $D_{0+}^{\nu}y(t)$ represents the Riemann–Liouville fractional derivative of order ν of the function $y(t)$. We also assume throughout that $n \in \mathbb{N}$ subject to the restriction that $n > 3$. The main contribution of this work is to determine conditions under which either problems (1.1)–(1.3) or (1.1), (1.4)–(1.6) will exhibit at least one positive solution. In particular, we shall state conditions on λ_1, λ_2, which are eigenvalues, for which problem (1.1)–(1.3) has at least one positive solution; it ought to be noted that *unlike in the integer-order case, the range of admissible eigenvalues depends on the choices of* ν_1, ν_2, *and* α. In addition, we state conditions on ϕ_1, ϕ_2 such that problem (1.1), (1.4)–(1.6) has at least one positive solution.

To place this problem in an appropriate context, we begin by remarking that Goodrich [1] considered a simpler version of problem (1.1)–(1.3). In particular, in [1] the problem

$$-D_{0+}^{\nu}y(t) = f(t, y(t)), \quad 0 < t < 1, \ n-1 < \nu \le n, \tag{1.7}$$

$$y^{(i)}(0) = 0, \quad 0 \le i \le n-2, \tag{1.8}$$

$$\left[D_{0+}^{\alpha}y(t)\right]_{t=1} = 0, \quad 1 \le \alpha \le n-2, \tag{1.9}$$

was considered. Obviously, this is closely related to our new problem (1.1)–(1.3), and, as a point of fact, we shall avail ourselves of certain of the results of this earlier paper. One of the reasons for considering this *particular* higher-dimensional problem is because of the nice properties of the Green's function for this problem, properties which are absent in certain other continuous fractional problems. (cf., Remark 2.9)

Somewhat more generally and also of great relevance, an integer-order version of (1.1) was considered recently by Henderson et al. [2]. In particular, they considered the problem

$$\begin{cases} u''(t) + \lambda a(t)f(v(t)) = 0, & 0 < t < 1, \\ v''(t) + \lambda b(t)f(u(t)) = 0, & 0 < t < 1, \end{cases} \tag{1.10}$$

subject to the multipoint-type boundary conditions

$$u(0) = \beta u(\eta), u(N) = \alpha u(\eta), v(0) = \beta v(\eta), v(N) = \alpha v(\eta). \tag{1.11}$$

They then deduced, under various assumptions on f, λ, a and b, the existence of at least one positive solution to (1.10)–(1.11). In particular, then, our results provide an immediate generalization of [2]. Moreover, our results here also generalize to some extent those of Zhang [3], who considered a problem similar to (1.7)–(1.9). Finally, we remark that there have been some attempts recently to address systems of fractional differential equations together with various boundary conditions. As a representative example, we remark that Su [4] considered the problem

$$\begin{cases} D^{\alpha}u(t) = f\left(t, v(t), D^{\mu}v(t)\right), & 0 < t < 1, \\ D^{\beta}v(t) = g\left(t, u(t), D^{\nu}u(t)\right), & 0 < t < 1, \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases} \tag{1.12}$$

where $1 < \alpha, \beta < 2$, $\mu, \nu > 0$, $\alpha - \nu \ge 1$, $\beta - \mu \ge 1$, and f, g are given functions.

More generally, the continuous fractional calculus has been studied extensively over the course of the past several decades. Some of this interest has derived from the fact that fractional differential equations provide, in certain cases, more realistic models for physical phenomena. In addition, however, the mathematical theory of the fractional calculus is itself highly nontrivial and, interestingly, certain aspects of the theory of ordinary and partial differential equations are not easily extended to the fractional setting. In particular, fractional differential equations, both ordinary and partial, have been studied from a variety of perspectives, including initial value problems, variational problems, delay and nonlocal equations, upper and lower solution techniques, and existence of positive solutions to boundary value problems—see, [5–19], for example, and the references therein. Other recent papers have also addressed fractional neutral differential equations [20,21] as well as fractional evolution equations and mild solutions [22,23]. It ought to be noted specifically that [9,11] address fractional boundary value problems with nonlocal boundary conditions, though the conditions we present in this work are different from the ones presented in [9,11].

We also ought to point out that fractional *difference* equations have also been studied more intensively of late. In particular, Atici and Eloe [24–28], Atici and Şengül [29], Goodrich [30–37], and Holm [38] have developed some of the theory of discrete fractional IVPs and BVPs as well as certain of the operational properties of the discrete fractional calculus with both delta and nabla derivatives. We note, in particular, that the problem considered in [34] is similar to the problem considered in this paper but in the discrete fractional setting, wherein some different techniques are required. In addition, a paper by Bastos et al. [39] has extended the discrete fractional calculus with delta derivative to the more general time scale $h\mathbb{Z}$, whereas papers by Anastassiou have provided some extension both the discrete fractional calculus with nabla derivative [40] and the discrete fractional calculus with delta derivative [41] to an arbitrary time scale. It turns out that the study of fractional difference equations can be quite delicate and mathematically nontrivial (cf., the previously cited works) especially when compared to the study of integer-order difference equations, and an increasing amount of research continues to unfold in this area.

With this context in mind, the outline of this paper is as follows. In Section 2 we shall recall certain results from the theory of the continuous fractional calculus. Moreover, we shall recall certain of the theorems deduced in [1]. In Section 3, we shall provide some conditions under which problem (1.1)–(1.3) will have at least one positive solution. In Section 4, we shall provide some alternative conditions under which problem (1.1), (1.4)–(1.6) will have at least one positive solution, including the special case in which $\lambda_1 = \lambda_2 = 1$. Our techniques in Sections 3 and 4 are cone theoretic in nature and are indebted to the classic paper of Erbe and Wang [42]. Finally, in Section 5, we shall provide some numerical examples, which shall explicate the applicability of our results.

2. Preliminaries

We first wish to collect some basic lemmas that will be important to us in the sequel. These and other related results and their proofs can be found, for example, in [8,16]. In particular, the excellent monograph by Podlubny [16] may be consulted for numerous other results concerning the continuous fractional calculus.

Definition 2.1. Let $\nu > 0$ with $\nu \in \mathbb{R}$. Suppose that $y : [a, +\infty) \to \mathbb{R}$. Then the ν-th Riemann–Liouville fractional integral is defined to be

$$D_{a^+}^{-\nu} y(t) := \frac{1}{\Gamma(\nu)} \int_a^t y(s)(t-s)^{\nu-1}\, \mathrm{d}s,$$

whenever the right-hand side is defined. Similarly, with $\nu > 0$ and $\nu \in \mathbb{R}$, we define the ν-th Riemann–Liouville fractional derivative to be

$$D_{a^+}^{\nu} y(t) := \frac{1}{\Gamma(n-\nu)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{y(s)}{(t-s)^{\nu+1-n}}\, \mathrm{d}s,$$

where $n \in \mathbb{N}$ is the unique positive integer satisfying $n-1 \le \nu < n$ and $t > a$.

Remark 2.2. In the sequel, we shall usually suppress the explicit dependence of $D_{a^+}^{\nu}$ on a. It will be clear from the context. In fact, in this paper $a = 0$ throughout.

Lemma 2.3. *Let $\alpha \in \mathbb{R}$. Then $D^n D^\alpha y(t) = D^{n+\alpha} y(t)$, for each $n \in \mathbb{N}_0$, where $y(t)$ is assumed to be sufficiently regular so that both sides of the equality are well-defined. Moreover, if $\beta \in (-\infty, 0]$ and $\gamma \in [0, +\infty)$, then $D^\gamma D^\beta y(t) = D^{\gamma+\beta} y(t)$.*

Lemma 2.4. *The general solution to $D^\nu y(t) = 0$, where $n-1 < \nu \le n$ and $\nu > 0$, is the function $y(t) = c_1 t^{\nu-1} + c_2 t^{\nu-2} + \cdots + c_n t^{\nu-n}$, where $c_i \in \mathbb{R}$ for each i.*

We now recall some results from [1] that will be of use in the sequel.

Theorem 2.5 ([1]). *Let $g \in \mathcal{C}([0, 1])$ be given. Then the unique solution to problem $-D^\nu y(t) = g(t)$ together with the boundary conditions $y^{(i)}(0) = 0 = \left[D_{0^+}^{\alpha} y(t)\right]_{t=1}$, where $1 \le \alpha \le n-2$ and $0 \le i \le n-2$, is*

$$y(t) = \int_0^1 G(t, s) g(s)\, \mathrm{d}s, \tag{2.1}$$

where

$$G(t, s) = \begin{cases} \dfrac{t^{\nu-1}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-1}}{\Gamma(\nu)}, & 0 \le s \le t \le 1 \\[2ex] \dfrac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}, & 0 \le t \le s \le 1 \end{cases} \tag{2.2}$$

is the Green's function for this problem.

Theorem 2.6 ([1]). *Let $G(t, s)$ be as given in the statement of Theorem 2.5. Then we find that:*

1. *$G(t, s)$ is a continuous function on the unit square $[0, 1] \times [0, 1]$;*
2. *$G(t, s) \ge 0$ for each $(t, s) \in [0, 1] \times [0, 1]$; and*
3. *$\max_{t \in [0,1]} G(t, s) = G(1, s)$, for each $s \in [0, 1]$.*

Theorem 2.7 ([1]). *Let $G(t,s)$ be as given in the statement of Theorem 2.5. Then there exists a constant $\gamma \in (0,1)$ such that*

$$\min_{t\in\left[\frac{1}{2},1\right]} G(t,s) \geq \gamma \max_{t\in[0,1]} G(t,s) = \gamma G(1,s). \tag{2.3}$$

We next recall the Krasnosel'skiĭ fixed point theorem—see, for example, [43]. This lemma will be of use in Sections 3 and 4 of this paper.

Lemma 2.8. *Let $\mathscr{B}$ be a Banach space and let $\mathscr{K} \subseteq \mathscr{B}$ be a cone. Assume that Ω_1 and Ω_2 are open sets contained in $\mathscr{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume, further, that $T : \mathscr{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \mathscr{K}$ is a completely continuous operator. If either*

1. *$\|Ty\| \leq \|y\|$ for $y \in \mathscr{K} \cap \partial\Omega_1$ and $\|Ty\| \geq \|y\|$ for $y \in \mathscr{K} \cap \partial\Omega_2$; or*
2. *$\|Ty\| \geq \|y\|$ for $y \in \mathscr{K} \cap \partial\Omega_1$ and $\|Ty\| \leq \|y\|$ for $y \in \mathscr{K} \cap \partial\Omega_2$;*

then T has at least one fixed point in $\mathscr{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

We conclude this section with a remark.

Remark 2.9. That the Green's function $G(t,s)$ given in Theorem 2.5 satisfies the Harnack-like inequality of Theorem 2.7 is crucial in the sequel. Interestingly, this sort of inequality does not necessarily hold in the fractional calculus—cf., [8]. In the *discrete* fractional calculus, however, this property does seem to hold always—see, for instance, [28,31,37].

3. Existence of a positive solution: case-I

In this section we wish to present a general condition under which problems (1.1)–(1.3) will exhibit at least one positive solution. We first need to fix our framework for analyzing problem (1.1)–(1.3).

First of all, let $\mathscr{B}$ represent the Banach space of $\mathcal{C}([0,1])$ when equipped with the usual supremum norm, $\|\cdot\|$. Then put

$$\mathscr{X} := \mathscr{B} \times \mathscr{B}, \tag{3.1}$$

where $\mathscr{X}$ is equipped with the norm

$$\|(y_1,y_2)\| := \|y_1\| + \|y_2\|, \tag{3.2}$$

for $(y_1,y_2) \in \mathscr{X}$. Observe that $\mathscr{X}$ is also a Banach space—see [44]. In addition, define the operators $T_1, T_2 : \mathscr{X} \to \mathscr{B}$ by

$$(T_1(y_1,y_2))(t) := \lambda_1 \int_0^1 G_1(t,s)a_1(s)f(y_1(s),y_2(s))\,\mathrm{d}s \tag{3.3}$$

and

$$(T_2(y_1,y_2))(t) := \lambda_2 \int_0^1 G_2(t,s)a_2(s)g(y_1(s),y_2(s))\,\mathrm{d}s, \tag{3.4}$$

where $G_1(t,s)$ is the Green's function of Theorem 2.5 with ν replaced by ν_1 and, likewise, $G_2(t,s)$ is the Green's function of Theorem 2.5 with ν replaced by ν_2. Using (3.3)–(3.4), define an operator $S : \mathscr{X} \to \mathscr{X}$ by

$$\begin{aligned}(S(y_1,y_2))(t) &:= ((T_1(y_1,y_2))(t), (T_2(y_1,y_2))(t))\\ &= \left(\lambda_1 \int_0^1 G_1(t,s)a_1(s)f(y_1(s),y_2(s))\,\mathrm{d}s, \lambda_2 \int_0^1 G_2(t,s)a_2(s)g(y_1(s),y_2(s))\,\mathrm{d}s\right).\end{aligned} \tag{3.5}$$

We claim that whenever $(y_1,y_2) \in \mathscr{X}$ is a fixed point of the operator defined in (3.5), it follows that $y_1(t)$ and $y_2(t)$ solve problems (1.1)–(1.3). This is the content of Lemma 3.1, whose proof we provide next.

Lemma 3.1. *A pair of functions $(y_1,y_2) \in \mathscr{X}$ is a solution of problems (1.1)–(1.3) if and only if (y_1,y_2) is a fixed point of the operator S defined in (3.5).*

Proof. The forward implication is immediate, owing to the result given in Theorem 2.5. Conversely, suppose that $(y_1,y_2) \in \mathscr{X}$ is a fixed point of the operator S. Then, in particular, we find that

$$y_1(t) = \lambda_1 \int_0^1 G_1(t,s)a_1(s)f(y_1(s),y_2(s))\,\mathrm{d}s. \tag{3.6}$$

Observe that the right-hand side of (3.6) may be recast as

$$\lambda_1 t^{\nu_1-1} \cdot \frac{\Gamma(\nu_1-\alpha)}{\Gamma(\nu_1)} \left[D^{\alpha-\nu_1}a_1(t)f(y_1(t),y_2(t))\right]_{t=1} - \lambda_1 D^{-\nu_1}\left[a_1(t)f(y_1(t),y_2(t))\right] \tag{3.7}$$

so that, in fact,

$$y_1(t)=\lambda_1 t^{\nu_1-1}\cdot\frac{\Gamma(\nu_1-\alpha)}{\Gamma(\nu_1)}\left[D^{\alpha-\nu_1}a_1(t)f(y_1(t),y_2(t))\right]_{t=1}-\lambda_1 D^{-\nu_1}\left[a_1(t)f(y_1(t),y_2(t))\right]. \tag{3.8}$$

We claim that $y_1(t)$ satisfies the differential equation (1.1) and the boundary conditions (1.2)–(1.3). To see that the former holds, apply the differential operator D^{ν_1} to both sides of (3.8) and recall (cf., [16]) that $D^{\nu_1}\left[t^{\nu_1-j}\right]=0$, for $1\le j\le n$, and that $D^{\nu_1}D^{-\nu_1}=D^0$. Then we find that

$$\begin{aligned}D^{\nu_1}y_1(t) &= \lambda_1 D^{\nu_1}\left[t^{\nu_1-1}\right]\cdot\frac{\Gamma(\nu_1-\alpha)}{\Gamma(\nu_1)}\left[D^{\alpha-\nu_1}a_1(t)f(y_1(t),y_2(t))\right]_{t=1}-\lambda_1 D^{\nu_1}\left[D^{-\nu_1}\left[a_1(t)f(y_1(t),y_2(t))\right]\right]\\ &= -\lambda_1 a_1(t)f(y_1(t),y_2(t)),\end{aligned} \tag{3.9}$$

from which we see that $y_1(t)$ satisfies the differential equation in (1.1). On the other hand, to see that $y_1(t)$ satisfies the boundary conditions in (1.2)–(1.3), fix an i satisfying $0\le i\le n-2$ and note that

$$\begin{aligned}y_1^{(i)}(t) &= \lambda_1(\nu_1-1)\cdots(\nu_1-i)\,t^{\nu_1-1-i}\cdot\frac{\Gamma(\nu_1-\alpha)}{\Gamma(\nu_1)}\left[D^{\alpha-\nu_1}a_1(t)f(y_1(t),y_2(t))\right]_{t=1}\\ &\quad-\lambda_1 D^iD^{-\nu_1}\left[a_1(t)f(y_1(t),y_2(t))\right].\end{aligned} \tag{3.10}$$

Recalling that $D^iD^{-\nu_1}=D^{i-\nu_1}$ (cf., Lemma 2.3) and that $\nu_1-1-i>0$, we find that

$$y_1^{(i)}(0)=\lambda_1\left\{0-D^{i-\nu_1}\left[a_1(t)f(y_1(t),y_2(t))\right]\right\}_{t=0}=0, \tag{3.11}$$

so that y_1 satisfies boundary condition (1.2). (Note that we have used the continuity of a_1 and f here so that $\left\{D^{i-\nu_1}\left[a_1(t)f(y_1(t),y_2(t))\right]\right\}_{t=0}$ has value 0.) Finally, recall that (see [16])

$$D^\alpha t^{\nu_1-1}=\frac{\Gamma(\nu_1)}{\Gamma(\nu_1-\alpha)}t^{\nu_1-\alpha-1}. \tag{3.12}$$

Then (3.12) together with an application of Lemma 2.3 implies that

$$\begin{aligned}D^\alpha y_1(t) &= \lambda_1 t^{\nu_1-\alpha-1}\left[D^{\alpha-\nu_1}a_1(t)f(y_1(t),y_2(t))\right]_{t=1}-\lambda_1 D^\alpha D^{-\nu_1}\left[a_1(t)f(y_1(t),y_2(t))\right]\\ &= \lambda_1 t^{\nu_1-\alpha-1}\left[D^{\alpha-\nu_1}a_1(t)f(y_1(t),y_2(t))\right]_{t=1}-\lambda_1 D^{\alpha-\nu_1}\left[a_1(t)f(y_1(t),y_2(t))\right]\end{aligned} \tag{3.13}$$

so that, since $\nu_1-\alpha-1>0$,

$$\begin{aligned}\left[D^\alpha y_1(t)\right]_{t=1} &= \lambda_1\left[D^{\alpha-\nu_1}a_1(t)f(y_1(t),y_2(t))\right]_{t=1}-\lambda_1 D^{\alpha-\nu_1}\left[a_1(t)f(y_1(t),y_2(t))\right]_{t=1}\\ &= 0,\end{aligned} \tag{3.14}$$

whence y_1 satisfies the boundary condition (1.3).

Now, a completely dual calculation reveals that y_2 also satisfies boundary conditions (1.2)–(1.3) and the differential equation $-D^{\nu_2}y_2(t)=\lambda_2 a_2(t)g(y_1(t),y_2(t))$. Therefore, we conclude that if $(y_1,y_2)\in\mathscr{X}$ is a fixed point of the operator S, then (y_1,y_2) solves the problems (1.1)–(1.3). And this completes the proof. □

As a consequence of Lemma 3.1, we shall look for fixed points of the operator S, seeing as these fixed points coincide with solutions of problems (1.1)–(1.3). For use in the sequel, let γ_1 and γ_2 the constants given by Theorem 2.7 associated, respectively, to the Green's functions G_1 and G_2, and define $\widetilde{\gamma}$ by

$$\widetilde{\gamma}:=\min\{\gamma_1,\gamma_2\}, \tag{3.15}$$

and notice that $\widetilde{\gamma}\in(0,1)$. Let us next introduce some conditions on the nonlinearities as well as the eigenvalues. These are very similar to those presented by Henderson et al. [2].

F1: There exist numbers f^* and g^*, with $f^*,g^*\in(0,+\infty)$, such that

$$\lim_{(y_1,y_2)\to(0^+,0^+)}\frac{f(y_1,y_2)}{y_1+y_2}=f^*\quad\text{and}\quad\lim_{(y_1,y_2)\to(0^+,0^+)}\frac{g(y_1,y_2)}{y_1+y_2}=g^*.$$

F2: There exist numbers f^{**} and g^{**}, with $f^{**},g^{**}\in(0,+\infty)$, such that

$$\lim_{(y_1,y_2)\to(\infty,\infty)}\frac{f(y_1,y_2)}{y_1+y_2}=f^{**}\quad\text{and}\quad\lim_{(y_1,y_2)\to(\infty,\infty)}\frac{g(y_1,y_2)}{y_1+y_2}=g^{**}.$$

L1: There are numbers Λ_1 and Λ_2, where

$$\Lambda_1 := \max\left\{\frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \widetilde{\gamma} G_1(1,s)\, a_1(s) f^{**}\, \mathrm{d}s\right]^{-1}, \frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \widetilde{\gamma} G_2(1,s)\, a_2(s) g^{**}\, \mathrm{d}s\right]^{-1}\right\} \tag{3.16}$$

and

$$\Lambda_2 := \min\left\{\frac{1}{2}\left[\int_{0}^{1} G_1(1,s) a_1(s) f^{*}\, \mathrm{d}s\right]^{-1}, \frac{1}{2}\left[\int_{0}^{1} G_2(1,s) a_2(s) g^{*}\, \mathrm{d}s\right]^{-1}\right\}, \tag{3.17}$$

such that $\Lambda_1 < \lambda_1, \lambda_2 < \Lambda_2$.

Next define the cone $\mathcal{K}$ by

$$\mathcal{K} := \left\{(y_1, y_2) \in \mathcal{X} : y_1, y_2 \geq 0,\ \min_{t\in\left[\frac{1}{2},1\right]} [y_1(t) + y_2(t)] \geq \widetilde{\gamma} \| (y_1, y_2) \|\right\}. \tag{3.18}$$

We then deduce the following lemma.

Lemma 3.2. *Let S be the operator defined by (3.5). Then $S : \mathcal{K} \to \mathcal{K}$.*

Proof. Let $(y_1, y_2) \in \mathcal{X}$ be given. It is clear from the definition of S together with the fact that a_1, a_2, f, and g are nonnegative that $T_1(y_1, y_2)(t) \geq 0$ and $T_2(y_1, y_2)(t) \geq 0$, for each $t \in [0, 1]$. On the other hand, we observe that

$$\begin{aligned}
\min_{t\in\left[\frac{1}{2},1\right]} [T_1(y_1, y_2)(t) + T_2(y_1, y_2)(t)] &\geq \min_{t\in\left[\frac{1}{2},1\right]} T_1(y_1, y_2)(t) + \min_{t\in\left[\frac{1}{2},1\right]} T_2(y_1, y_2)(t) \\
&\geq \gamma_1 \|T_1(y_1, y_2)\| + \gamma_2 \|T_2(y_1, y_2)\| \\
&\geq \widetilde{\gamma}\, [\|T_1(y_1, y_2)\| + \|T_2(y_1, y_2)\|] \\
&= \widetilde{\gamma} \| (T_1(y_1, y_2), T_2(y_1, y_2)) \| \\
&= \widetilde{\gamma} \|S(y_1, y_2)\|.
\end{aligned} \tag{3.19}$$

So, we conclude that $S : \mathcal{K} \to \mathcal{K}$, as desired. And this completes the proof. □

We now state our existence theorem. While this theorem is similar to the existence theorem provided in [2], it is completely new in the fractional-order case. Moreover, in Section 4 we shall give results that more substantially generalize even the integer-order results presented in [2].

Theorem 3.3. *Suppose that conditions* (F1)–(F2) *and* (L1) *are satisfied. Then problem (1.1)–(1.3) has at least one positive solution.*

Proof. We have already shown in Lemma 3.2 that $S : \mathcal{K} \to \mathcal{K}$. Furthermore, a relatively straightforward application of the Arzela-Ascoil theorem, which we omit, reveals that S is a completely continuous operator.

Now, observe that by condition (L1) that there is $\epsilon > 0$ sufficiently small such that

$$\max\left\{\frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \widetilde{\gamma} G_1(1,s)\, a_1(s) \left(f^{**} - \epsilon\right)\, \mathrm{d}s\right]^{-1}, \frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \widetilde{\gamma} G_2(1,s)\, a_2(s) \left(g^{**} - \epsilon\right)\, \mathrm{d}s\right]^{-1}\right\} \leq \lambda_1, \lambda_2 \tag{3.20}$$

and

$$\lambda_1, \lambda_2 \leq \min\left\{\frac{1}{2}\left[\int_{0}^{1} G_1(1,s) a_1(s) \left(f^{*} + \epsilon\right)\, \mathrm{d}s\right]^{-1}, \frac{1}{2}\left[\int_{0}^{1} G_2(1,s) a_2(s) \left(g^{*} + \epsilon\right)\, \mathrm{d}s\right]^{-1}\right\}. \tag{3.21}$$

Now, given this ϵ, by condition (F1) it follows that there exists some number $r_1^{*} > 0$ such that

$$f(y_1, y_2) \leq \left(f^{*} + \epsilon\right)(y_1 + y_2), \tag{3.22}$$

whenever $\| (y_1, y_2) \| < r_1^{*}$. Similarly, by condition (F1), for the same ϵ, there exists a number $r_1^{**} > 0$ such that

$$g(y_1, y_2) \leq \left(g^{*} + \epsilon\right)(y_1 + y_2), \tag{3.23}$$

whenever $\| (y_1, y_2) \| < r_1^{**}$. In particular, then, by putting $r_1 := \min\left\{r_1^{*}, r_1^{**}\right\}$, we find that both (3.22) and (3.23) hold whenever $\| (y_1, y_2) \| < r_1$. So, define Ω_1 by

$$\Omega_1 := \{(y_1, y_2) \in \mathcal{X} : \| (y_1, y_2) \| < r_1\}. \tag{3.24}$$

Then for $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_1$ we find that

$$\begin{aligned}
\|T_1(y_1, y_2)\| &\leq \lambda_1 \int_0^1 G_1(1, s) a_1(s) f(y_1(s), y_2(s))\, ds \\
&\leq \lambda_1 \int_0^1 G_1(1, s) a_1(s) \left(f^* + \epsilon\right)(y_1(s) + y_2(s))\, ds \\
&\leq \lambda_1 \int_0^1 G_1(1, s) a_1(s) \left(f^* + \epsilon\right)(\|y_1\| + \|y_2\|)\, ds \\
&= \|(y_1, y_2)\| \cdot \lambda_1 \int_0^1 G_1(1, s) a_1(s) \left(f^* + \epsilon\right) ds \\
&\leq \frac{1}{2}\|(y_1, y_2)\|. \qquad (3.25)
\end{aligned}$$

We may deduce by an entirely dual argument that

$$\|T_2(y_1, y_2)\| \leq \frac{1}{2}\|(y_1, y_2)\|. \qquad (3.26)$$

Thus, by putting (3.22)–(3.26) together we find that for $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_1$ we have

$$\begin{aligned}
\|S(y_1, y_2)\| &= \|(T_1(y_1, y_2), T_2(y_1, y_2))\| = \|T_1(y_1, y_2)\| + \|T_2(y_1, y_2)\| \\
&\leq \frac{1}{2}\|(y_1, y_2)\| + \frac{1}{2}\|(y_1, y_2)\| = \|(y_1, y_2)\|, \qquad (3.27)
\end{aligned}$$

so that S is a cone compression on $\mathcal{K} \cap \partial\Omega_1$.

On the other hand, letting ϵ be the same positive number selected at the beginning of this proof, note that by virtue of condition (F2) we can find a number $\widetilde{r}_2 > 0$ such that

$$f(y_1, y_2) \geq \left(f^{**} - \epsilon\right)(y_1 + y_2) \qquad (3.28)$$

and

$$g(y_1, y_2) \geq \left(g^{**} - \epsilon\right)(y_1 + y_2), \qquad (3.29)$$

whenever $y_1 + y_2 \geq \widetilde{r}_2$. Put

$$r_2 := \max\left\{2r_1, \frac{\widetilde{r}_2}{\widetilde{\gamma}}\right\}. \qquad (3.30)$$

Moreover, put

$$\Omega_2 := \{(y_1, y_2) \in \mathscr{X} : \|(y_1, y_2)\| < r_2\}. \qquad (3.31)$$

Note that if $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_2$, then it follows that, for any $t \in \left[\frac{1}{2}, 1\right]$,

$$y_1(t) + y_2(t) \geq \min_{t \in \left[\frac{1}{2}, 1\right]} [y_1(t) + y_2(t)] \geq \widetilde{\gamma}\|(y_1, y_2)\| \geq \widetilde{r}_2. \qquad (3.32)$$

In particular, (3.32) shows that whenever $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_2$, it follows that $\|(y_1, y_2)\| \geq \widetilde{r}_2$ so that (3.28)–(3.29) hold. Then for each $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_2$ we find that

$$\begin{aligned}
T_1(y_1, y_2)(1) &= \lambda_1 \int_0^1 G_1(1, s) a_1(s) f(y_1(s), y_2(s))\, ds \\
&\geq \lambda_1 \int_{\frac{1}{2}}^1 G_1(1, s) a_1(s) f(y_1(s), y_2(s))\, ds \\
&\geq \lambda_1 \int_{\frac{1}{2}}^1 G_1(1, s) a_1(s) \left(f^{**} - \epsilon\right)(y_1(s) + y_2(s))\, ds \\
&\geq \lambda_1 \int_{\frac{1}{2}}^1 \widetilde{\gamma} G_1(1, s) a_1(s) \left(f^{**} - \epsilon\right)\|(y_1, y_2)\|\, ds. \qquad (3.33)
\end{aligned}$$

Thus, we conclude from (3.33) that

$$\|T_1\left(y_1, y_2\right)\| \geq \frac{1}{2}\|\left(y_1, y_2\right)\|, \tag{3.34}$$

whenever $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_2$. Similarly, we find that

$$\|T_2\left(y_1, y_2\right)\| \geq \frac{1}{2}\|\left(y_1, y_2\right)\|. \tag{3.35}$$

Consequently, (3.28)–(3.35) imply that

$$\begin{aligned}\|S\left(y_1, y_2\right)\| &= \|\left(T_1\left(y_1, y_2\right), T_2\left(y_1, y_2\right)\right)\| = \|T_1\left(y_1, y_2\right)\| + \|T_2\left(y_1, y_2\right)\| \\ &\geq \frac{1}{2}\|\left(y_1, y_2\right)\| + \frac{1}{2}\|\left(y_1, y_2\right)\| = \|\left(y_1, y_2\right)\|,\end{aligned} \tag{3.36}$$

whenever $(y_1, y_2) \in \mathcal{K} \cap \partial\Omega_2$. Thus, S is a cone expansion on $\mathcal{K} \cap \partial\Omega_2$.

In summary, each of the hypotheses of Lemma 2.8 is satisfied. Consequently, we conclude that S has a fixed point, say $\left(y_1^0, y_2^0\right) \in \mathcal{K}$. As the pair of functions $y_1^0(t), y_2^0(t)$ is a positive solution of (1.1)–(1.3), the theorem is proved. □

We conclude this section with a remark about Theorem 3.3.

Remark 3.4. Evidently, by choosing t differently in (3.33), we would obtain a slightly different form for Λ_1. However, the form given in (3.16), which is the one induced by the choice of $t = 1$ in (3.33), is the optimal choice as it minimizes the value of Λ_1.

4. Existence of a positive solution: case-II

We now wish to provide a set of conditions under which problem (1.1), (1.4)–(1.6) will have at least one positive solution. In particular, we shall consider two such cases. As remarked in Section 1, we note that although the boundary conditions in (1.5) and (1.6) do not necessarily possess any physical meaning when $\alpha \notin \mathbb{N}$, they are mathematically meaningful. Moreover, in case α is an integer, then these boundary conditions become physically meaningful.

4.1. Problem (1.1), (1.4)–(1.6) in the general case

In this subsection, we consider the general problem (1.1), (1.4)–(1.6) in the sense that λ_1, λ_2 can range over a continuum of values, which we shall specify presently. We shall still need conditions (F1)–(F2) in this setting. However, because the boundary conditions are now given by (1.4)–(1.6), we shall introduce a new condition, labeled (G1) in the sequel. Condition (G1) provides some control over the nonlocal boundary terms, ϕ_1 and ϕ_2. We state this condition now and then give a remark explicating the form and nature of these nonlocal functionals.

G1: The functionals $\phi_1(y_1)$ and $\phi_2(y_2)$ are continuous in y_1 and y_2, nonnegative for $y_1, y_2 \geq 0$, and satisfy

$$\lim_{\|y\|\to 0^+} \frac{\phi_1(y_1)}{\|y_1\|} = 0 \tag{4.1}$$

and

$$\lim_{\|y\|\to 0^+} \frac{\phi_2(y_2)}{\|y_2\|} = 0, \tag{4.2}$$

respectively.

L2: There are numbers Λ_3 and Λ_4, where

$$\Lambda_3 := \max\left\{\frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \gamma_0 G_1\left(1, s\right) a_1(s) f^{**}\, \mathrm{d}s\right]^{-1}, \frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \gamma_0 G_2\left(1, s\right) a_2(s) g^{**}\, \mathrm{d}s\right]^{-1}\right\}, \tag{4.3}$$

$$\Lambda_4 := \min\left\{p\left[\int_0^1 G_1(1, s) a_1(s) f^{*}\, \mathrm{d}s\right]^{-1}, p\left[\int_0^1 G_2(1, s) a_2(s) g^{*}\, \mathrm{d}s\right]^{-1}\right\}, \tag{4.4}$$

and $p \in \left(0, \frac{1}{2}\right)$ is given, such that $\Lambda_3 < \lambda_1, \lambda_2 < \Lambda_4$ and where γ_0 is the constant defined in (4.16) in the sequel.

Remark 4.1. Let us make some additional comments regarding condition (G1) above. First of all, we interpret these limits in the sense that (4.1) is true only if for each $\eta > 0$ there is $r > 0$ such that whenever $0 < \|y_1\| \leq r$, it follows that $0 \leq \frac{\phi_1(y_1)}{\|y_1\|} < \eta$. The same may be said of condition (4.2) involving ϕ_2.

Second of all, let us explicitly point out that this condition is indeed satisfied by nontrivial functionals $\phi : \mathcal{C}([0,1]) \to \mathbb{R}$. For instance, consider the functional

$$\phi_1(y) := [y(t_0)]^{\gamma}, \tag{4.5}$$

where $\phi_1 : \mathcal{C}([0,1]) \to \mathbb{R}$ and $t_0 \in (0,1)$, $\gamma > 1$ are given. Let $\eta > 0$ be given. Then for nonnegative y, we find that whenever $0 < \|y\| \le \eta^{\frac{1}{\gamma-1}}$, it follows that

$$0 \le \frac{\phi_1(y)}{\|y\|} \le \frac{\|y\|^{\gamma}}{\|y\|} = \|y\|^{\gamma-1} \le \left(\eta^{\frac{1}{\gamma-1}}\right)^{\gamma-1} = \eta,$$

so that the condition described in the preceding paragraph is satisfied—note that we chose $r := \eta^{\frac{1}{\gamma-1}} > 0$ here.

We present now a trio of preliminary lemmas. These shall also be of use in Section 4.2 in the sequel.

Lemma 4.2. *A pair of functions $(y_1, y_2) \in \mathscr{X}$ is a solution of (1.1), (1.4)–(1.6) if and only if (y_1, y_2) is a fixed point of the operator $U : \mathscr{X} \to \mathscr{X}$ defined by*

$$\begin{aligned}(U(y_1,y_2))(t) := \Bigg(&\beta_1(t)\phi_1(y_1) + \lambda_1 \int_0^1 G_1(t,s)a_1(s)f(y_1(s), y_2(s))\, ds,\\ &\beta_2(t)\phi_2(y_2) + \lambda_2 \int_0^1 G_2(t,s)a_2(s)g(y_1(s), y_2(s))\, ds\Bigg),\end{aligned} \tag{4.6}$$

where $\beta_1, \beta_2 : [0,1] \to [0,1]$ are defined by

$$\beta_1(t) := \frac{\Gamma(\nu_1 - \alpha)}{\Gamma(\nu_1)} t^{\nu_1 - 1} \tag{4.7}$$

and

$$\beta_2(t) := \frac{\Gamma(\nu_2 - \alpha)}{\Gamma(\nu_2)} t^{\nu_2 - 1}. \tag{4.8}$$

Proof. To prove this lemma, we can essentially repeat the proof of Lemma 3.1 given earlier together with a minor modification of the proof of Theorem 2.5 presented in [1]. Indeed, define $U_1, U_2 : \mathscr{X} \to \mathscr{B}$ by, say,

$$U_1(y_1,y_2)(t) := \beta_1(t)\phi_1(y_1) + \lambda_1 \int_0^1 G_1(t,s)a_1(s)f(y_1(s), y_2(s))\, ds \tag{4.9}$$

and

$$U_2(y_1,y_2)(t) := \beta_2(t)\phi_2(y_2) + \lambda_2 \int_0^1 G_2(t,s)a_2(s)g(y_1(s), y_2(s))\, ds. \tag{4.10}$$

A verification very similar to the proof of Lemma 3.1 reveals that

$$\left(U_j^{(i)}(y_1,y_2)\right)(0) = 0, \tag{4.11}$$

for each $0 \le i \le n-2$ and each $j = 1, 2$, and that

$$D_{0^+}^{\alpha}\left[U_j(y_1,y_2)\right]_{t=1} = \phi_j(y_j), \tag{4.12}$$

for each $j = 1, 2$. Moreover, we find that, for each $j = 1, 2$, the operator $U_j(y_1,y_2)(t)$ satisfies the j-th equation in (1.1). Therefore, it follows that if $(y_1,y_2) \in \mathscr{X}$ is a fixed point of the operator U defined in (4.4), then the pair of functions $y_1(t), y_2(t)$ is a solution of the boundary value problem (1.1), (1.4)–(1.6). And this completes the proof. □

Lemma 4.3. *Let $\beta_1(t)$ and $\beta_2(t)$ be defined as in (4.7) and (4.8) above. Then each of $\beta_1(t)$ and $\beta_2(t)$ is strictly increasing in t and satisfy $\beta_1(0) = \beta_2(0) = 0$ and $\beta_1(1), \beta_2(1) \in (0,1)$. Moreover, there exist constants M_{β_1} and M_{β_2} satisfying $M_{\beta_1}, M_{\beta_2} \in (0,1)$ such that $\min_{t\in\left[\frac{1}{2},1\right]} \beta_1(t) \ge M_{\beta_1}\|\beta_1\|$ and $\min_{t\in\left[\frac{1}{2},1\right]} \beta_2(t) \ge M_{\beta_2}\|\beta_2\|$.*

Proof. It is obvious that $\beta_1(0) = \beta_2(0) = 0$. Moreover, since $\nu_1, \nu_2 > 1$, it is also obvious that both β_1 and β_2 are strictly increasing for $t \in [0,1]$. Moreover, recall that $\nu_1, \nu_2 > 3$. Then as both $\nu_1 - \alpha \ge 1$ and $\nu_2 - \alpha \ge 1$, it follows $0 < \frac{\Gamma(\nu_i - \alpha)}{\Gamma(\nu_i)} < 1$, for each $i = 1, 2$. Finally, from the preceding properties, the final statement in the lemma is obviously true. And this completes the proof. □

Remark 4.4. Let us note at this juncture that

$$M_{\beta_1} := M_{\beta_1}(\nu_1) = \left(\frac{1}{2}\right)^{\nu_1-1} \tag{4.13}$$

and that

$$M_{\beta_2} := M_{\beta_2}(\nu_2) = \left(\frac{1}{2}\right)^{\nu_2-1}, \tag{4.14}$$

which may be easily verified by simply observing, for instance, that $M_{\beta_1} = \frac{\beta_1\left(\frac{1}{2}\right)}{\beta_1(1)}$.

In light of Lemma 4.3, let us define a new cone $\mathcal{K}_1$ by

$$\mathcal{K}_1 := \left\{(y_1, y_2) \in \mathscr{X} : y_1, y_2 \geq 0, \min_{t\in\left[\frac{1}{2},1\right]} [y_1(t) + y_2(t)] \geq \gamma_0 \|(y_1, y_2)\|\right\}, \tag{4.15}$$

where

$$\gamma_0 := \min\left\{\widetilde{\gamma}, M_{\beta_1}, M_{\beta_2}\right\}. \tag{4.16}$$

It is obvious that $\gamma_0 \in (0, 1)$.

Lemma 4.5. *Let U be the operator defined in (4.6). Then $U : \mathcal{K}_1 \to \mathcal{K}_1$.*

Proof. Let U_1 and U_2 be defined as in (4.9) and (4.10), respectively, above. Then whenever $(y_1, y_2) \in \mathcal{K}_1$, it is clear that $U_1(y_1, y_2)(t), U_2(y_1, y_2)(t) \geq 0$, for each $t \in [0, 1]$. On the other hand, in light of Lemma 4.3 and the definition of γ_0 provided in (4.16), we find that

$$\begin{aligned}
\min_{t\in\left[\frac{1}{2},1\right]} [U_1(y_1, y_2)(t) + U_2(y_1, y_2)(t)] &\geq \min_{t\in\left[\frac{1}{2},1\right]} \beta_1(t)\phi_1(y_1) + \min_{t\in\left[\frac{1}{2},1\right]} \lambda_1 \int_0^1 G_1(t, s)a_1(s)f(y_1(s), y_2(s))\,ds \\
&\quad + \min_{t\in\left[\frac{1}{2},1\right]} \beta_2(t)\phi_2(y_2) + \min_{t\in\left[\frac{1}{2},1\right]} \lambda_2 \int_0^1 G_2(t, s)a_2(s)g(y_1(s), y_2(s))\,ds \\
&\geq M_{\beta_1} \max_{t\in[0,1]} \beta_1(t)\phi_1(y_1) + \gamma_1 \max_{t\in[0,1]} \lambda_1 \int_0^1 G_1(t, s)a_1(s)f(y_1(s), y_2(s))\,ds \\
&\quad + M_{\beta_2} \max_{t\in[0,1]} \beta_2(t)\phi_2(y_2) + \gamma_2 \max_{t\in[0,1]} \lambda_2 \int_0^1 G_2(t, s)a_2(s)g(y_1(s), y_2(s))\,ds \\
&\geq \gamma_0 \max_{t\in[0,1]} \left[\beta_1(t)\phi_1(y_1) + \lambda_1 \int_0^1 G_1(t, s)a_1(s)f(y_1(s), y_2(s))\,ds\right] \\
&\quad + \gamma_0 \max_{t\in[0,1]} \left[\beta_2(t)\phi_2(y_2) + \lambda_2 \int_0^1 G_2(t, s)a_2(s)g(y_1(s), y_2(s))\,ds\right] \\
&= \gamma_0\|U_1(y_1, y_2)\| + \gamma_0\|U_2(y_1, y_2)\| \\
&= \gamma_0\|(U_1(y_1, y_2), U_2(y_1, y_2))\| \\
&= \gamma_0\|U(y_1, y_2)\|,
\end{aligned} \tag{4.17}$$

whence

$$\min_{t\in\left[\frac{1}{2},1\right]} [U_1(y_1, y_2)(t) + U_2(y_1, y_2)(t)] \geq \gamma_0\|(U_1(y_1, y_2), U_2(y_1, y_2))\|, \tag{4.18}$$

as desired. Thus, we conclude that $U : \mathcal{K}_1 \to \mathcal{K}_1$, as claimed. And this completes the proof. □

We are now ready to state and prove our first existence theorem for problem (1.1), (1.4)–(1.6).

Theorem 4.6. *Suppose that conditions* (F1)–(F2), (G1), *and* (L2) *hold. Then problem (1.1), (1.4)–(1.6) has at least one positive solution.*

Proof. Lemma 4.5 shows that $U : \mathcal{K}_1 \to \mathcal{K}_1$. Moreover, due to the continuity of β_1, β_2, ϕ_1, and ϕ_2, it is clear that both U_1 and U_2 are completely continuous operators by a standard application of the Arzela–Ascoli theorem, which we omit.

Let p be the given number satisfying $0 < p < \frac{1}{2}$, as in the statement of condition (L2) above. Now, just as in the proof of Theorem 3.3, there is by condition (L2) a number $\epsilon > 0$ such that

$$\Lambda_3 := \max\left\{\frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \gamma_0 G_1(1,s)\,a_1(s)\left(f^{**}-\epsilon\right)\,ds\right]^{-1}, \frac{1}{2}\left[\int_{\frac{1}{2}}^{1} \gamma_0 G_2(1,s)\,a_2(s)\left(g^{**}-\epsilon\right)\,ds\right]^{-1}\right\} \le \lambda_1, \lambda_2 \tag{4.19}$$

and

$$\lambda_1, \lambda_2 \le \min\left\{p\left[\int_0^1 G_1(1,s)a_1(s)\left(f^{*}+\epsilon\right)\,ds\right]^{-1}, p\left[\int_0^1 G_2(1,s)a_2(s)\left(g^{*}+\epsilon\right)\,ds\right]^{-1}\right\}. \tag{4.20}$$

Given this ϵ, just as before, conditions (3.22) and (3.23) remain true whenever $\|(y_1,y_2)\| < r_1$, exactly as in the proof of Theorem 3.3. In this case, however, we need to use condition (G1) as well to further refine the choice of r_1. In particular, by condition (G1) it follows that there is a number, say, $r_1^{***} > 0$ such that $\phi(y_1) \le \eta\|y_1\|$ whenever $0 < \|y_1\| \le r_1^{***}$. In particular and without loss of generality, let us suppose that $0 < \eta_1 < \frac{1}{2} - p$. (Note that by the choice of p, we clearly have that $\frac{1}{2} - p > 0$.) Now, put $\widetilde{r}_1 := \min\left\{r_1, r_1^{***}\right\}$. Then we find for all $(y_1,y_2) \in \mathscr{X}$ satisfying $0 < \|(y_1,y_2)\| < \widetilde{r}_1$ both that

$$\begin{cases} f(y_1,y_2) \le \left(f^{*}+\epsilon\right)(y_1+y_2) \\ g(y_1,y_2) \le \left(g^{*}+\epsilon\right)(y_1+y_2) \end{cases} \tag{4.21}$$

and that

$$\phi(y_1) \le \left(\frac{1}{2} - p\right)\|y_1\|. \tag{4.22}$$

So, define Ω_1 by $\Omega_1 := \{(y_1,y_2) \in \mathscr{X} : \|(y_1,y_2)\| < \widetilde{r}_1\}$. Observe that for any $(y_1,y_2) \in \mathcal{K}$ we have that $\|y_1\|, \|y_2\| \le \|(y_1,y_2)\|$. We then find for $(y_1,y_2) \in \mathcal{K}_1 \cap \Omega_1$ that

$$\begin{aligned} \|U_1(y_1,y_2)\| &\le \phi_1(y_1) + \lambda_1\int_0^1 G_1(1,s)a_1(s)f(y_1(s),y_2(s))\,ds \\ &\le \left(\frac{1}{2}-p\right)\|y_1\| + \lambda_1\int_0^1 G_1(1,s)a_1(s)f(y_1(s),y_2(s))\,ds \\ &\le \left(\frac{1}{2}-p\right)\|(y_1,y_2)\| + \lambda_1\int_0^1 G_1(1,s)a_1(s)\left(f^{*}+\epsilon\right)(y_1(s)+y_2(s))\,ds \\ &\le \|(y_1,y_2)\|\left[\left(\frac{1}{2}-p\right) + \lambda_1\int_0^1 G_1(1,s)a_1(s)\left(f^{*}-\epsilon\right)\,ds\right] \\ &\le \|(y_1,y_2)\|\left[\left(\frac{1}{2}-p\right)+p\right] \\ &\le \frac{1}{2}\|(y_1,y_2)\|, \end{aligned} \tag{4.23}$$

whence $\|U_1(y_1,y_2)\| \le \frac{1}{2}\|(y_1,y_2)\|$. A similar analysis shows that $\|U_2(y_1,y_2)\| \le \frac{1}{2}\|(y_1,y_2)\|$. Consequently, we conclude that whenever $(y_1,y_2) \in \mathcal{K}_1 \cap \partial\Omega_1$, it follows that

$$\|U(y_1,y_2)\| \le \|(y_1,y_2)\| \tag{4.24}$$

so that U is a cone compression on $\mathcal{K}_1 \cap \partial\Omega_1$.

Conversely, let ϵ be the same number selected at the beginning of this proof. As before, condition (F2) implies the existence of a number r_2^{*} such that

$$f(y_1,y_2) \ge \left(f^{**}-\epsilon\right)(y_1+y_2) \tag{4.25}$$

and

$$g(y_1,y_2) \ge \left(g^{**}-\epsilon\right)(y_1+y_2) \tag{4.26}$$

whenever $y_1 + y_2 \ge r_2^{*}$. In addition, recall that by condition (G1) it follows that ϕ_1 and ϕ_2 are assumed to be nonnegative for $(y_1,y_2) \in \mathcal{K}_1$. Finally, if we put

$$r_2 := \max\left\{2r_1, \frac{r_2^{*}}{\gamma_0}\right\}, \tag{4.27}$$

similar to (3.30) earlier, then it follows that a condition like (3.32) holds whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$, where we put

$$\Omega_2 := \{(y_1, y_2) \in \mathscr{X} : \| (y_1, y_2) \| < r_2\}. \tag{4.28}$$

Thus, for each $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$, it follows that

$$\begin{aligned} U_1 (y_1, y_2)(1) &\geq \lambda_1 \int_0^1 G_1(1, s)a_1(s)f(y_1(s), y_2(s))\, ds \\ &\geq \lambda_1 \int_{\frac{1}{2}}^1 \gamma_0 G_1(1, s)a_1(s)\left(f^{**} - \epsilon\right) \| (y_1, y_2) \| \, ds \\ &\geq \frac{1}{2} \| (y_1, y_2) \|, \end{aligned} \tag{4.29}$$

where we have used the nonnegativity of ϕ_1 to get the first inequality in (4.29). Consequently, (4.29) implies that $\|U_1 (y_1, y_2)\| \geq \frac{1}{2} \| (y_1, y_2) \|$ whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$. A similar calculation reveals that $\|U_2 (y_1, y_2)\| \geq \frac{1}{2} \| (y_1, y_2) \|$ whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$. Thus, we conclude that

$$\|U (y_1, y_2)\| \geq \| (y_1, y_2) \|, \tag{4.30}$$

whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$.

Finally, combining (4.24) and (4.30) and applying Lemma 2.8, we find that there exists a fixed point $\left(y_1^0, y_2^0\right) \in \mathscr{X}$ of the operator U. As the pair of functions $y_1^0(t), y_2^0(t)$ is a solution of problem (1.1), (1.4)–(1.6), the proof is complete. □

Remark 4.7. Observe that the eigenvalue problem considered by Theorem 4.6 could not be handled (even in the *integer-order case*) by the results of Henderson et al. [2]. Thus, Theorem 4.6 is an essential generalization of problem (1.1) not only in the fractional-order case but also in the integer-order case.

4.2. Problem (1.1), (1.4)–(1.6) in case $\lambda_1 = \lambda_2 = 1$

In contrast to the previous subsection, we now specialize to the case in which $\lambda_1 = \lambda_2 = 1$. In this case, problem (1.1), (1.4)–(1.6) is no longer an eigenvalue problem. Consequently, we shall no longer have any need to invoke condition (L2). Furthermore, we shall alter conditions (F1)–(F2) since their imposition was a consequence of condition (L2). In particular, then, we begin by introducing the following new conditions. Note that we retain condition (G1) as before, and so, we shall not list it separately below.

F3: We find that

$$\lim_{(y_1,y_2)\to(0^+,0^+)} \frac{f(y_1, y_2)}{y_1 + y_2} = 0 \quad \text{and} \quad \lim_{(y_1,y_2)\to(0^+,0^+)} \frac{g(y_1, y_2)}{y_1 + y_2} = 0.$$

F4: We find that

$$\lim_{(y_1,y_2)\to(\infty,\infty)} \frac{f(y_1, y_2)}{y_1 + y_2} = +\infty \quad \text{and} \quad \lim_{(y_1,y_2)\to(\infty,\infty)} \frac{g(y_1, y_2)}{y_1 + y_2} = +\infty.$$

We present now two preliminary lemmas. First, let us make a remark.

Remark 4.8. In the sequel, we shall represent by U^1 the operator U with $\lambda_1 = \lambda_2 = 1$. In addition, we shall represent by U_1^1 and U_2^1 the operators U_1 and U_2, respectively, with $\lambda_1 = \lambda_2 = 1$.

Lemma 4.9. *A pair of functions $(y_1, y_2) \in \mathscr{X}$ is a solution of (1.1), (1.4)–(1.6), in case $\lambda_1 = \lambda_2 = 1$, if and only if (y_1, y_2) is a fixed point of the operator U^1 defined by*

$$\begin{aligned} \left(U^1 (y_1, y_2)\right)(t) := \Bigg(&\beta_1(t)\phi_1(y_1) + \int_0^1 G_1(t, s)a_1(s)f(y_1(s), y_2(s))\, ds, \\ &\beta_2(t)\phi_2(y_2) + \int_0^1 G_2(t, s)a_2(s)g(y_1(s), y_2(s))\, ds \Bigg), \end{aligned} \tag{4.31}$$

where $\beta_1, \beta_2 : [0, 1] \to [0, 1]$ are defined by (4.7) and (4.8), respectively.

Proof. The proof of this lemma is essentially the same as the proof of Lemma 4.2. Consequently, we omit it. □

Lemma 4.10. *Let U^1 be the operator defined in (4.31). Then $U^1 : \mathcal{K}_1 \to \mathcal{K}_1$.*

Proof. Let U_1^1 and U_2^2 be defined as in Remark 4.8 above. Then whenever $(y_1, y_2) \in \mathcal{K}_1$, it is clear that $U_1^1(y_1, y_2)(t)$, $U_2^1(y_1, y_2)(t) \geq 0$, for each $t \in [0, 1]$.

On the other hand, in light of Lemma 4.3 and the definition of γ_0 provided in (4.16), we find that

$$\begin{aligned}\min_{t\in\left[\frac{1}{2},1\right]}\left[U_1^1(y_1,y_2)(t)+U_2^1(y_1,y_2)(t)\right] &\geq M_{\beta_1}\max_{t\in[0,1]}\beta_1(t)\phi_1(y_1)+\gamma\max_{t\in[0,1]}\int_0^1 G_1(t,s)a_1(s)f(y_1(s),y_2(s))\,ds\\ &\quad+M_{\beta_2}\max_{t\in[0,1]}\beta_2(t)\phi_2(y_2)+\gamma\max_{t\in[0,1]}\int_0^1 G_2(t,s)a_2(s)g(y_1(s),y_2(s))\,ds\\ &\geq \gamma_0\max_{t\in[0,1]}\left[\beta_1(t)\phi_1(y_1)+\int_0^1 G_1(t,s)a_1(s)f(y_1(s),y_2(s))\,ds\right]\\ &\quad+\gamma_0\max_{t\in[0,1]}\left[\beta_2(t)\phi_2(y_2)+\int_0^1 G_2(t,s)a_2(s)g(y_1(s),y_2(s))\,ds\right]\\ &=\gamma_0\|U^1(y_1,y_2)\|,\end{aligned}\tag{4.32}$$

whence

$$\min_{t\in\left[\frac{1}{2},1\right]}\left[U_1^1(y_1,y_2)(t)+U_2^1(y_1,y_2)(t)\right]\geq\gamma_0\|\left(U_1^1(y_1,y_2),U_2^1(y_1,y_2)\right)\|,\tag{4.33}$$

as desired. Thus, we conclude that $U^1:\mathcal{K}_1\to\mathcal{K}_1$, as claimed. And this completes the proof. □

We now present another existence theorem for problem (1.1), (1.4)–(1.6), this one in the special case when $\lambda_1=\lambda_2=1$.

Theorem 4.11. *Suppose that conditions* (F3)–(F4) *and* (G1) *hold. Then problem* (1.1), (1.4)–(1.6), *in the case where* $\lambda_1=\lambda_2=1$, *has at least one positive solution.*

Proof. Lemma 4.10 shows that $U^1:\mathcal{K}_1\to\mathcal{K}_1$. Moreover, due to the continuity of β_1, β_2, ϕ_1, and ϕ_2, it is clear that both U_1^1 and U_2^1 are completely continuous operators by a standard application of the Arzela–Ascoli theorem, which we again omit.

On the other hand, choose a number $\eta_1>0$ such that

$$0<\eta_1\left[1+\int_0^1 G_1(1,s)a_1(s)\,ds\right]<\frac{1}{2}.\tag{4.34}$$

Due condition (F3), note that there is a number $r_1^*>0$ such that $f(y_1,y_2)\leq\eta_1[y_1+y_2]$ whenever $0<\|(y_1,y_2)\|\leq r_1^*$. In addition, letting η_1 be the same number, by condition (G1) it follows that there is a number $r_1^{**}>0$ such that $\phi_1(y_1)\leq\eta_1\|y_1\|$ whenever $0<\|y_1\|\leq r_1^{**}$. Now, take $r_1:=\min\left\{r_1^*,r_1^{**}\right\}$. Observe that whenever $0<\|(y_1,y_2)\|<r_1$, it follows that $\|y_1\|<r_1\leq r_1^{**}$. In particular, then, for all $(y_1,y_2)\in\mathscr{X}$ satisfying $0<\|(y_1,y_2)\|<r_1$, we find both that

$$f(y_1,y_2)\leq\eta_1[y_1+y_2]\tag{4.35}$$

and that

$$\phi_1(y_1)\leq\eta_1\|y_1\|\leq\eta_1\|(y_1,y_2)\|.\tag{4.36}$$

So, put $\Omega_1:=\{(y_1,y_2)\in\mathscr{X}:\|(y_1,y_2)\|<r_1\}$. Then from (4.34)–(4.36), we find whenever $(y_1,y_2)\in\mathcal{K}_1\cap\partial\Omega_1$ that

$$\begin{aligned}\|U_1^1(y_1,y_2)\| &\leq \|\beta_1\|\phi_1(y_1)+\max_{t\in[0,1]}\int_0^1 G_1(t,s)a_1(s)f(y_1(s),y_2(s))\,ds\\ &\leq\eta_1\|y_1\|+\int_0^1 G_1(1,s)a_1(s)\eta_1[y_1(s)+y_2(s)]\,ds\\ &\leq\eta_1\|(y_1,y_2)\|+\|(y_1,y_2)\|\int_0^1\eta_1 G_1(1,s)a_1(s)\,ds\\ &\leq\|(y_1,y_2)\|\cdot\eta_1\left[1+\int_0^1 G_1(1,s)a_1(s)\,ds\right]\\ &\leq\frac{1}{2}\|(y_1,y_2)\|,\end{aligned}\tag{4.37}$$

whence $\|U_1^1(y_1,y_2)\|\leq\frac{1}{2}\|(y_1,y_2)\|$. Similarly, it can be shown that

$$\|U_2^1(y_1,y_2)\|\leq\frac{1}{2}\|(y_1,y_2)\|,\tag{4.38}$$

whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_1$. Therefore, from (4.37)–(4.38) we conclude that whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_1$, it follows that

$$\|U^1(y_1, y_2)\| \le \|(y_1, y_2)\|. \tag{4.39}$$

Conversely, recall that by assumption (G1), we have that $\phi_1(y_1), \phi_2(y_2) \ge 0$, for each $(y_1, y_2) \in \mathcal{K}_1$ (because $y_1, y_2 \ge 0$ whenever $(y_1, y_2) \in \mathcal{K}_1$). In addition, choose a number $\eta_2 > 0$ such that

$$\eta_2 \int_{\frac{1}{2}}^{1} \gamma_0 G_1\left(\frac{3}{4}, s\right) a_1(s)\, ds \ge \frac{1}{2}. \tag{4.40}$$

Then condition (F4) implies the existence of a number $r_2^* > 0$ such that whenever $\|(y_1, y_2)\| \ge r_2^*$, we find that

$$f(y_1, y_2) \ge \eta_2 [y_1 + y_2]. \tag{4.41}$$

Now, put

$$r_2 := \max\left\{2r_1, \frac{r_2^*}{\gamma_0}\right\}, \tag{4.42}$$

and define the set Ω_2 by $\Omega_2 := \{(y_1, y_2) \in \mathcal{X} : \|(y_1, y_2)\| < r_2\}$. Then from (4.40)–(4.42), it follows that

$$\begin{aligned} U_1^1(y_1, y_2)\left(\frac{3}{4}\right) &\ge \int_0^1 G_1\left(\frac{3}{4}, s\right) a_1(s) f(y_1(s), y_2(s))\, ds \\ &\ge \int_{\frac{1}{2}}^1 G_1\left(\frac{3}{4}, s\right) a_1(s) f(y_1(s), y_2(s))\, ds \\ &\ge \int_{\frac{1}{2}}^1 G_1\left(\frac{3}{4}, s\right) a_1(s) \eta_2 [y_1(s) + y_2(s)]\, ds \\ &\ge \|(y_1, y_2)\| \cdot \eta_2 \int_{\frac{1}{2}}^1 \gamma_0 G_1\left(\frac{3}{4}, s\right) a_1(s)\, ds \\ &\ge \frac{1}{2}\|(y_1, y_2)\|, \end{aligned} \tag{4.43}$$

whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$. Thus, we conclude that for any $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_1$

$$\|U_1^2(y_1, y_2)\| \ge \frac{1}{2}\|(y_1, y_2)\|. \tag{4.44}$$

A completely similar calculation shows that

$$\|U_2^1(y_1, y_2)\| \ge \frac{1}{2}\|(y_1, y_2)\|, \tag{4.45}$$

whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$. Thus, combining (4.44)–(4.45) implies that

$$\|U^1(y_1, y_2)\| \ge \|(y_1, y_2)\|, \tag{4.46}$$

whenever $(y_1, y_2) \in \mathcal{K}_1 \cap \partial\Omega_2$.

Finally, combining (4.39) and (4.46) and applying Lemma 2.8, we find that there exists a fixed point $\left(y_1^0, y_2^0\right) \in \mathcal{X}$ of the operator U^1. As the pair of functions $y_1^0(t), y_2^0(t)$ is a solution of problem (1.1), (1.4)–(1.6), the proof is complete. □

Let us conclude this section with a final remark.

Remark 4.12. To the best of the author's knowledge, Theorems 4.6 and 4.11 provides new results not only for the fractional-order problem (1.1), (1.4)–(1.6), but also for the corresponding *integer-order* problem—i.e., in the case $\nu_1 = \nu_2$ with $\nu_1, \nu_2 \in \mathbb{N}$.

5. Numerical examples

We now present two numerical examples illustrating, respectively, Theorems 3.3 and 4.11.

Example 5.1. Consider the problem, for $t \in (0, 1)$,

$$\begin{cases} -D_{0^+}^{5.2} y_1(t) = 12.5e^{-2t}(y_1(t) + y_2(t))\left(20\,000 - \dfrac{19\,990}{(y_1(t))^2 + (y_2(t))^2 + 1}\right) \\ -D_{0^+}^{5.95} y_2(t) = 5.75e^{-3t}(y_1(t) + y_2(t))\left(30\,000 - \dfrac{29\,995}{(y_1(t))^2 + (y_2(t))^2 + 1}\right), \end{cases} \tag{5.1}$$

subject to the boundary conditions

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0), \quad 0 \le i \le 4 \tag{5.2}$$

and

$$D_{0^+}^{1.5}\left[y_1(t)\right]_{t=1} = 0 = D_{0^+}^{1.5}\left[y_2(t)\right]_{t=1}. \tag{5.3}$$

Obviously, problem (5.1)–(5.3) fits the framework of problem (1.1)–(1.3) with $\nu_1 := 5.2$, $\nu_2 := 5.95$, $\alpha = 1.5$, $\lambda_1 = 12.5$, and $\lambda_2 = 5.75$. (Note that $n = 6$, therefore, in this case.) In addition, we have set

$$f(y_1, y_2) := (y_1 + y_2)\left(20\,000 - \frac{19\,990}{y_1^2 + y_2^2 + 1}\right), \tag{5.4}$$

$$g(y_1, y_2) := (y_1 + y_2)\left(30\,000 - \frac{29\,995}{y_1^2 + y_2^2 + 1}\right), \tag{5.5}$$

$$a_1(t) := e^{-2t} \tag{5.6}$$

and

$$a_2(t) := e^{-3t}. \tag{5.7}$$

Note that $f, g : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ and are continuous. The functions $a_1(t)$ and $a_2(t)$ are obviously nonnegative.

We now check that each of the conditions of Theorem 3.3 holds. In particular, observe that

$$\lim_{(y_1,y_2)\to(0^+,0^+)} \frac{f(y_1, y_2)}{y_1 + y_2} = \lim_{(y_1,y_2)\to(0^+,0^+)} \left(20\,000 - \frac{19\,990}{(y_1(t))^2 + (y_2(t))^2 + 1}\right) = 10 \tag{5.8}$$

and that

$$\lim_{(y_1,y_2)\to(0^+,0^+)} \frac{g(y_1, y_2)}{y_1 + y_2} = \lim_{(y_1,y_2)\to(0^+,0^+)} \left(30\,000 - \frac{29\,995}{(y_1(t))^2 + (y_2(t))^2 + 1}\right) = 5. \tag{5.9}$$

Thus, put

$$f^* := 10 \tag{5.10}$$

and

$$g^* := 5. \tag{5.11}$$

On the other hand, observe that

$$\lim_{(y_1,y_2)\to(\infty,\infty)} \frac{f(y_1, y_2)}{y_1 + y_2} = \lim_{(y_1,y_2)\to(\infty,\infty)} \left(20\,000 - \frac{19\,990}{(y_1(t))^2 + (y_2(t))^2 + 1}\right) = 20\,000 \tag{5.12}$$

and that

$$\lim_{(y_1,y_2)\to(\infty,\infty)} \frac{g(y_1, y_2)}{y_1 + y_2} = \lim_{(y_1,y_2)\to(\infty,\infty)} \left(30\,000 - \frac{29\,995}{(y_1(t))^2 + (y_2(t))^2 + 1}\right) = 30\,000. \tag{5.13}$$

Thus, put

$$f^{**} := 20\,000 \tag{5.14}$$

and

$$g^{**} := 30\,000. \tag{5.15}$$

In summary, (5.8)–(5.15) show that conditions (F1) and (F2) hold, as desired.

On the other hand, to calculate the admissible range of the eigenvalues λ_1, λ_2, as given by condition (L1), observe by numerical approximation we find that

$$\Lambda_1 \approx 5.451 \tag{5.16}$$

and that

$$\Lambda_2 \approx 38.717. \tag{5.17}$$

Thus, for any λ_1, λ_2 satisfying

$$5.451 < \lambda_1, \qquad \lambda_2 < 38.717 \tag{5.18}$$

condition (L1) will be satisfied. Since it is clear from (5.1) that

$$\lambda_1, \lambda_2 \in [5.451, 38.717], \tag{5.19}$$

we find that condition (L1) is satisfied.

Thus, we see that each of conditions (F1)–(F2) and (L1) is satisfied. Consequently, (5.4)–(5.19) imply by Theorem 3.3 that problem (5.1)–(5.3) has at least one positive solution.

Example 5.2. Consider the problem, for $t \in (0, 1)$,

$$\begin{cases} -D_{0^+}^{7.52} y_1(t) = \mathrm{e}^{-2t}\left[y_1^2 + y_2^2\right] \\ -D_{0^+}^{7.31} y_2(t) = \mathrm{e}^{-3t}\left[y_1^3 + y_2^2\right], \end{cases} \tag{5.20}$$

subject to the boundary conditions

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0), \quad 0 \le i \le 6 \tag{5.21}$$

and

$$\begin{cases} D_{0^+}^{2.25}\left[y_1(t)\right]_{t=1} = \left[y_1\left(\dfrac{1}{2}\right)\right]^6 \\ D_{0^+}^{2.25}\left[y_2(t)\right]_{t=1} = \left[y_2\left(\dfrac{3}{4}\right)\right]^{\frac{3}{2}}. \end{cases} \tag{5.22}$$

Obviously, problem (5.20)–(5.22) fits the framework of boundary value problem (1.1), (1.4)–(1.6). In particular, boundary condition (5.22) represents a nonlocal condition. Note that in this case we have selected $\nu_1 := 7.52$, $\nu_2 := 7.31$, and $\alpha = 2.25$; it is also the case that $n = 8$ here. Furthermore, we have that

$$f(y_1, y_2) := y_1^2 + y_2^2, \tag{5.23}$$

$$g(y_1, y_2) := y_1^3 + y_2^2, \tag{5.24}$$

$$a_1(t) := \mathrm{e}^{-2t}, \tag{5.25}$$

$$a_2(t) := \mathrm{e}^{-3t}, \tag{5.26}$$

$$\phi_1(y_1) := \left[y_1\left(\frac{1}{2}\right)\right]^6 \tag{5.27}$$

and

$$\phi_2(y_2) := \left[y_2\left(\frac{3}{4}\right)\right]^{\frac{3}{2}}. \tag{5.28}$$

We check that conditions (F3)–(F4) and (G1) hold. In particular, observe that

$$\lim_{(y_1,y_2)\to(0^+,0^+)} \frac{y_1^2 + y_2^2}{y_1 + y_2} = 0, \tag{5.29}$$

$$\lim_{(y_1,y_2)\to(0^+,0^+)} \frac{y_1^3 + y_2^2}{y_1 + y_2} = 0, \tag{5.30}$$

$$\lim_{(y_1,y_2)\to(\infty,\infty)} \frac{y_1^2 + y_2^2}{y_1 + y_2} = +\infty, \tag{5.31}$$

$$\lim_{(y_1,y_2)\to(\infty,\infty)} \frac{y_1^3 + y_2^2}{y_1 + y_2} = +\infty, \tag{5.32}$$

so that conditions (F3)–(F4) are seen to hold. On the other hand, note that

$$0 \le \lim_{\|y_1\|\to 0^+} \frac{\phi_1(y_1)}{\|y_1\|} = \lim_{\|y_1\|\to 0^+} \frac{\left[y_1\left(\frac{1}{2}\right)\right]^6}{\|y_1\|} \le \lim_{\|y_1\|\to 0^+} \frac{\|y_1\|^6}{\|y_1\|} = \lim_{\|y_1\|\to 0^+} \|y_1\|^5 = 0 \tag{5.33}$$

and, similarly, that

$$0 \leq \lim_{\|y_2\| \to 0^+} \frac{\phi_2(y_2)}{\|y_2\|} = \lim_{\|y_2\| \to 0^+} \frac{\left[y_2\left(\frac{3}{4}\right)\right]^{\frac{3}{2}}}{\|y_2\|} \leq \lim_{\|y_2\| \to 0^+} \frac{\|y_2\|^{\frac{3}{2}}}{\|y_2\|} = \lim_{\|y_2\| \to 0^+} \|y_2\|^{\frac{1}{2}} = 0, \tag{5.34}$$

whence by (5.33) and (5.34), respectively, we find that condition (G1) holds, too.

Thus, conditions (F3)–(F4) and (G1) hold. Therefore, by (5.23)–(5.34) together with Theorem 4.11 we conclude that problem (5.20)–(5.22) has at least one positive solution, as desired.

Remark 5.3. As implied elsewhere, to the best of the author's knowledge, the problems in Examples 5.1 and 5.2 could not be handled by other results presently in the literature. In particular, these examples show how our results here extend those presented in [1–4,8], for example.

Remark 5.4. Observe that the orders of the fractional derivatives, namely ν_1, ν_2, and α, affect the admissible range of eigenvalues both in (3.16)–(3.17) and in (4.3)–(4.4). *Thus, in the problems considered here, we really have three extra parameters affecting the problem than in the corresponding integer-order problem*.

Remark 5.5. It should be noted that in approximating the admissible range of eigenvalues in (5.19), we used the fact, which is established in [1], that

$$\gamma := \min\left\{\frac{\left(\frac{1}{2}\right)^{\nu-\alpha-1}}{2^\alpha - 1}, \left(\frac{1}{2}\right)^{\nu-1}\right\}.$$

We conclude with two remarks regarding classes of functions satisfying conditions (F1)–(F2).

Remark 5.6. One fairly broad class of (nontrivial) functions satisfying conditions (F1)–(F2) are given by

$$f(\mathbf{x}) := C_1 e^{-g(\mathbf{x})} \nabla \cdot \mathbf{H}(\mathbf{x}),$$

where $g : \overline{\mathbb{R}^n_+} \to [0, +\infty)$, $f : \overline{\mathbb{R}^n_+} \to [0, +\infty)$, $C_1 > 0$ is a constant, $\mathbf{H} : \overline{\mathbb{R}^n_+} \to \overline{\mathbb{R}^n_+}$ is the vector field defined by

$$\mathbf{H}(\mathbf{x}) := \sum_{i=1}^{n} \frac{1}{2} x_i^2 \mathbf{e}_i,$$

where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{R}^n$, and by $\overline{\mathbb{R}^n_+}$ we mean the closure of the interior of the positive cone in $\mathbb{R}^n$. Obviously the class of functions $L(y_1, y_2) = ay_1 + ay_2$ trivially satisfies (F1)–(F2), for $a > 0$.

Remark 5.7. Another class of (nontrivial) functions satisfying conditions (F1)–(F2) is

$$f(x, y) := (x + y)\left(A + \frac{B}{x^2 + y^2 + C}\right), \tag{5.35}$$

for appropriately chosen constants $A, C \in [0, +\infty)$ and $B \in \mathbb{R}$ subject to the stipulation that $f : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous. Obviously the choice of function in Example 5.1 fits the framework of (5.35).

6. Conclusions

In this paper we have considered a continuous boundary value problem of fractional-order. In particular, using ideas established by Goodrich [1] and Henderson et al. [2] together with some ideas of Erbe and Wang [42] and Dunninger and Wang [44] we have demonstrated that both problem (1.1)–(1.3) and (1.1), (1.4)–(1.6) can exhibit at least one positive solution under various assumptions on both the nonlinearities, the nonlocal boundary terms (if present), and the eigenvalues (if present). Our results here extend work by Bai and Lü [8], Goodrich [1], Henderson et al. [2], and Zhang [3], among others.

As a means of concluding this work, let us suggest three possible avenues for additional study of problem (1.1).

1. One potentially interesting avenue for additional research on problem (1.1) might be to allow the nonlinearities f and g to depend on (fractional) derivatives of y_1 and y_2, say in a manner similar to problem (1.12) considered by Su [4]. While our work does extend [4], it does not extend the full problem considered in [4], for our nonlinearities do not depend on any derivatives (integer-order or otherwise) of the unknown functions y_1 and y_2. This could be an interesting direction in which to study this problem. Moreover, to the best of the author's knowledge, that sort of generalization has not been considered in the discrete fractional case either.
2. Another possible avenue of study might be to investigate the effect of more specific, nonlocal conditions and their effect on the admissible range of eigenvalues. Here we have considered rather general nonlocal conditions, but it could be interesting to investigate the effect of very specific forms of the nonlocal functionals ϕ_1 and ϕ_2.

3. While we have pointed out (cf., Remark 5.4) that the admissible range of eigenvalues depend explicitly upon the choices of ν_1, ν_2, and α, it seems that by means of numerical simulations, these relationships could be further explored and clarified. These relationships could be interesting to investigate and would possibly provide insight beyond what has been provided in the present work.

In any case, there seem to be many avenues for addition study of this problem. We hope that as the continuous and discrete fractional calculus continue to evolve and mature additional work will be completed on this and related problems.

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